Differential Geometry Chapter 2

Differentiable maps

We examine maps $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ for $U \subseteq \mathbb{R}^n$ an open subset. Let f^j, $1 \leq j \leq m$ be the coordinate functions of $\mathbf{f}$.

In this course we do not look at the largest class of differentiable functions, i.e. Frechet differentiable. Instead we restrict to

Definition 1 $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *is a* C^∞ *or* ***smooth differentiable*** *map if all the partial derivatives of all orders of all* f^i *exist and are continuous on* U.

Lemma 2 ***Chain Rule*** *Let* $\mathbf{g} : W \subseteq \mathbb{R}^p \to \mathbb{R}^n$, *with* $\mathbf{g}(W) \subseteq U$, *and* $\mathbf{f} : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ *be differentiable functions.*

Let $\mathbf{h} = \mathbf{f} \circ \mathbf{g} : W \subseteq \mathbb{R}^p \to \mathbb{R}^m$.

Let y^j, $1 \leq j \leq p$ *be the variables in* $\mathbb{R}^p$ *and* x^i, $1 \leq i \leq n$ *the variables in* $\mathbb{R}^n$. *Then* $\mathbf{h}$ *is differentiable on* W *and*

$$\frac{\partial \mathbf{h}}{\partial y^j}(\mathbf{y}) = \sum_{i=1}^{n} \frac{\partial \mathbf{f}}{\partial x^i}(g(\mathbf{y})) \frac{\partial g^i}{\partial y^j}(\mathbf{y})$$

for all $1 \leq j \leq p$ *and* $\mathbf{y} \in W$.

Proof not given. See Calculus of Several Variables.

Example $p = 1$, and $m = 1$. So $h = f \circ \mathbf{g} : \mathbb{R} \to \mathbb{R}$. Then

$$\frac{dh}{dt}(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\mathbf{g}(t)) \frac{dg^i}{dt}(t). \tag{1}$$

Definition 3 *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be a differentiable map and* $\mathbf{v_p}$ *a tangent vector to* $\mathbb{R}^n$. *Then the* ***derivative of*** f ***with respect to*** $\mathbf{v_p}$ *is*

$$\mathbf{v_p}[f] = \frac{d}{dt} f(\mathbf{p} + t\mathbf{v})_{t=0}.$$

This was known as $d_{\mathbf{v}} f(\mathbf{p})$ in my Calculus of Several Variables course; the directional derivative at $\mathbf{p}$ in the direction $\mathbf{v}$. (I restricted, though, to unit $\mathbf{v}$).

If we choose $\mathbf{g}(t) = \mathbf{p} + t\mathbf{v}$ in (1) we see that

$$\frac{d}{dt} f(\mathbf{p} + t\mathbf{v}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\mathbf{p} + t\mathbf{v})\, v^i,$$

so

$$\mathbf{v}_{\mathbf{p}}[f] = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\mathbf{p})\, v^i.$$

This would appear to be a dot product of $\mathbf{v}_{\mathbf{p}}$ with a vector with components $\partial f(\mathbf{p}) / \partial x^i$. This second vector is called the gradient vector:

Definition 4 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable map. The* ***gradient vector*** *of f at $\mathbf{p}$ is the tangent vector*

$$\nabla f(\mathbf{p}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\mathbf{p})\, U_i(\mathbf{p}).$$

Thus

$$\mathbf{v}_{\mathbf{p}}[f] = \nabla f(\mathbf{p}) \bullet \mathbf{v}_{\mathbf{p}}.$$

Since $f \mapsto \nabla f(\mathbf{p})$ is a linear operator with

$$\nabla (fg)(\mathbf{p}) = \nabla f(\mathbf{p})\, g(\mathbf{p}) + f(\mathbf{p})\, \nabla g(\mathbf{p}),$$

for $f, g : \mathbb{R}^n \to \mathbb{R}$ differentiable maps, the following follows quickly.

Lemma 5 *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be differentiable maps; $\mathbf{v}_{\mathbf{p}}$, $\mathbf{w}_{\mathbf{p}}$ tangent vectors and $a, b \in \mathbb{R}$. Then*

i. $(a\,\mathbf{v}_{\mathbf{p}} + b\,\mathbf{w}_{\mathbf{p}})[f] = a\,\mathbf{v}_{\mathbf{p}}[f] + b\,\mathbf{w}_{\mathbf{p}}[f]$,

ii. $\mathbf{v}_{\mathbf{p}}[af + bg] = a\,\mathbf{v}_{\mathbf{p}}[f] + b\,\mathbf{v}_{\mathbf{p}}[g]$,

iii. $\mathbf{v}_{\mathbf{p}}[fg] = \mathbf{v}_{\mathbf{p}}[f]\, g(\mathbf{p}) + \mathbf{v}_{\mathbf{p}}[g]\, f(\mathbf{p})$.

Proof left as an exercise. ■

Curves in $\mathbb{R}^n$

Definition 6 *A* ***curve in*** $\mathbb{R}^n$ *is a differentiable map* $\alpha : I \to \mathbb{R}^n$*, from an interval* $I \subseteq \mathbb{R}$*.*

Example 7 $\alpha(t) = \mathbf{p} + t\mathbf{v}$ *is a straight line through* $\mathbf{p}$ *in the* $\mathbf{v}$ *direction,*
$\alpha(t) = (a\cos t,\, a\sin t,\, 0)$ *is a circle in the* x*-*y *plane, though it is a curve in* $\mathbb{R}^3$*,*
$\alpha(t) = (a\cos t,\, a\sin t,\, bt)$ *is a (right-hand) helix in* $\mathbb{R}^3$*.*

Definition 8 *Let* $\alpha : I \to \mathbb{R}^n$ *be a curve in* $\mathbb{R}^n$*. The* ***velocity vector of*** α *at* t *is the tangent vector*

$$\alpha'(t) = (\alpha_1'(t), \ldots, \alpha_n'(t))^T_{\alpha(t)}.$$

Note that $\alpha'(t)$ is a vector field on the curve, i.e. to every point on the curve, $\alpha(t)$, it associates a tangent vector $\alpha'(t)$.

Question Let f be a differentiable map $f : \mathbb{R}^n \to \mathbb{R}$ and α a curve in $\mathbb{R}^n$. What is the rate of change of f along α?

Lemma 9 *With the notation as above*

$$\frac{df(\alpha(t))}{dt} = \alpha'(t)[f].$$

Proof by (1) above,

$$\frac{df(\alpha(t))}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\boldsymbol{\alpha}(t)) \frac{\partial \alpha_i}{\partial t}(t).$$

Comparing with

$$\mathbf{v}_{\mathbf{p}}[f] = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(\mathbf{p})\, v^i$$

gives our result. ■

Assume that if V a vector field on $\mathbb{R}^n$, written as $\sum_{i=1}^n f_i U_i$, then each $f_i : \mathbb{R}^n \to \mathbb{R}$ is a differentiable map.

Given a vector field V and a path α consider

$$V(\alpha(t)) = \sum_{i=1}^{n} f_i(\alpha(t))\, U_i(\alpha(t)).$$

Definition 10 *We define the derivative V' at a point on the curve by*

$$\begin{aligned} V'(\alpha(t)) &= \sum_{i=1}^{n} \frac{d}{dt} f_i(\alpha(t))\, U_i\left(\alpha\left(t\right)\right) \\ &= \sum_{i=1}^{n} \alpha'\left(t\right)[f_i]\ U_i\left(\alpha\left(t\right)\right) \end{aligned}$$

by previous lemma.

The derivative $V'$is a vector field defined on the curve and represents the rate of change of $V(\mathbf{p})$ as $\mathbf{p}$ goes along the curve.

Special Case 1. If $V(\alpha(t)) = \alpha'(t)$, (so this vector field is defined only on the curve, not all of $\mathbb{R}^n$), then

$$\alpha''(t) = \sum_{i=1}^{n} \frac{d^2}{dt^2} \alpha_i(t)\, U_i\left(\alpha(t)\right).$$

This is the **acceleration** of the curve at $\alpha\left(t\right)$.

Special Case 2. If $V = \sum_{i=1}^{n} f_i\, U_i$ is general but with the specific $\alpha\left(t\right) = \mathbf{p} + t\mathbf{v}$ so $\alpha'(0) = \mathbf{v_p}$. Then

$$V'\left(\mathbf{p} + t\mathbf{v}\right)\big|_{t=0} = \sum_{i=1}^{n} \alpha'(0)[f_i]\ U_i(\alpha(0)).$$

Lemma 11 *Let $U = \sum_{i=1}^{n} u_i\, U_i$, $V = \sum_{i=1}^{n} v_i\, U_i$ be vector fields; α a curve and write $U(t) = U(\alpha(t))$ etc. Then*

i. $(\lambda U + \mu V)' = \lambda U' + \mu V'$ for all $\lambda, \mu \in \mathbb{R}$,
ii. $(U \bullet V)' = U' \bullet V + U \bullet V'$,
iii. If $U \bullet V : I \to \mathbb{R}$ is constant then $U' \bullet V + U \bullet V' = 0$,
iv. For differentiable $f : I \to \mathbb{R}$

$$(fU)' = \frac{df}{dt} U + fU'.$$

Proof Exercise. ■

This derivative of a vector field along a curve only depends on the initial velocity of the curve, a tangent vector. So we could take this as the definition of

Definition 12 *The **covariant derivative** of V w.r.t. $\mathbf{v_p}$ is*

$$\nabla_{\mathbf{v_p}} V = V'(\mathbf{p} + t\mathbf{v})|_{t=0} = \sum_{i=1}^{n} \mathbf{v_p}[f_i] \, U_i(\mathbf{p}) \, .$$

This measures the initial rate of change of $V(\mathbf{p})$ as $\mathbf{p}$ moves in the $\mathbf{v}$ direction.

Lemma 13 *Let $U = \sum_{i=1}^{n} u_i \, U_i$, $V = \sum_{i=1}^{n} v_i \, U_i$ be vector fields and $\mathbf{u_p}$, $\mathbf{v_p}$ tangent vectors. Then*

i. for all $\lambda, \mu \in \mathbb{R}$,

$$\nabla_{\lambda \mathbf{u_p} + \mu \mathbf{v_p}} U = \lambda \nabla_{\mathbf{u_p}} U + \mu\lambda \nabla_{\mathbf{v_p}} U,$$

ii. for all $\lambda, \mu \in R$,

$$\nabla_{\mathbf{v_p}} (\lambda U + \mu V) = \lambda \nabla_{\mathbf{v_p}} U + \mu \nabla_{\mathbf{v_p}} V,$$

iii. For differentiable $f : I \to \mathbb{R}$

$$\nabla_{\mathbf{v_p}} (fU) = \mathbf{v_p} [f] \, U(\mathbf{p}) + f(\mathbf{p}) \, \nabla_{\mathbf{v_p}} (U) \, ,$$

iv.

$$\mathbf{v_p} [U \bullet V] = \nabla_{\mathbf{v_p}} U \bullet V(\mathbf{p}) + U(\mathbf{p}) \bullet \nabla_{\mathbf{v_p}} V.$$

Proof Exercise. For example

iii. The Right Hand Side equals

$$\begin{aligned}
& \sum_{i=1}^{n} \mathbf{v_p}[u_i] \, U_i(\mathbf{p}) \bullet \sum_{i=1}^{n} v_i(\mathbf{p}) \, U_i(\mathbf{p}) + \sum_{i=1}^{n} u_i(\mathbf{p}) \, U_i(\mathbf{p}) \bullet \sum_{i=1}^{n} \mathbf{v_p}[v_i] \, U_i(\mathbf{p}) \\
= & \sum_{i=1}^{n} \left(\mathbf{v_p}[u_i] \; v_i(\mathbf{p}) + u_i(\mathbf{p}) \; \mathbf{v_p}[v_i] \right) \\
= & \sum_{i=1}^{n} \mathbf{v_p}[u_i v_i] = \mathbf{v_p} \left[\sum_{i=1}^{n} u_i \, v_i \right] \\
= & \; \mathbf{v_p}[U \bullet V] \, .
\end{aligned}$$

■

The generalising to Vector Fields can continue. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and V a vector field on $\mathbb{R}^n$.

Definition 14 *Define $V[f]$ to be a function $\mathbb{R}^n \to \mathbb{R}$ such that*

$$V[f](\mathbf{p}) = V(\mathbf{p})[f],$$

for all $\mathbf{p} \in \mathbb{R}^n$.

From $\mathbf{v_p}[f] = \nabla f(\mathbf{p}) \bullet \mathbf{v_p}$ we see that $V[f](\mathbf{p}) = \nabla f(\mathbf{p}) \bullet V(\mathbf{p})$, the component of the gradient vector at $\mathbf{p}$ in the direction of $V(\mathbf{p})$. We could thus write $V[f] = \nabla f \bullet V$.

This makes it easy to prove

$$V[fg] = V[f]\, g + fV[g], \tag{2}$$

for example.

The last definition has taken the previously defined $\mathbf{v_p}[f]$ to defined $V[f]$. Similarly we can take the previously defined $\nabla_{\mathbf{v_p}} W$ to define $\nabla_V W$:

Definition 15 *If V, W are vector fields on $\mathbb{R}^n$ then $\nabla_V W$ is a vector field on $\mathbb{R}^n$ such that*

$$\nabla_V W(\mathbf{p}) = \nabla_{V(\mathbf{p})} W$$

for all $\mathbf{p} \in \mathbb{R}^n$.

By the definition $\nabla_{\mathbf{v_p}} V = \sum_{i=1}^n \mathbf{v_p}[f_i]\ U_i(p)$ we have

$$\nabla_V W(\mathbf{p}) = \nabla_{V(\mathbf{p})} W = \sum_{i=1}^n V(\mathbf{p})[w_i]\ U_i(\mathbf{p}) = \sum_{i=1}^n V[w_i](\mathbf{p})\ U_i(\mathbf{p}),$$

having used the previous definition. Thus we can write

$$\nabla_V W = \sum_{i=1}^n V[w_i]\ U_i.$$

Then, for differentiable $f : \mathbb{R}^n \to \mathbb{R}$ result (2) gives

$$\begin{aligned} \nabla_V (fW) &= \sum_{i=1}^n (fV[w_i] + w_i V[f])\, U_i \\ &= \nabla_V (W)\, f + V[f]\, W. \end{aligned}$$

Or, for two vector fields $U = \sum_{i=1}^{n} u_i U_i$ and $W = \sum_{i=1}^{n} w_i U_i$ on $\mathbb{R}^n$,

$$\begin{aligned} V[U \bullet W] &= V\left[\sum_{i=1}^{n} u_i w_i\right] \\ &= \sum_{i=1}^{n} V[u_i w_i] \quad \text{since } V \text{ is linear, see } () \,. \\ &= \sum_{i=1}^{n} (u_i V[w_i] + w_i V[u_i]) \quad \text{by } () \\ &= U \bullet \nabla_V (W) + W \bullet \nabla_V (U)\,. \end{aligned}$$

Recap

For $\mathbf{v_p} \in T(\mathbb{R}^3)$ and differentiable $f : \mathbb{R}^3 \to \mathbb{R}$ we define $\mathbf{v_p}[f] = f'(\mathbf{p} + t\mathbf{v})|_{t=0} \in \mathbb{R}$.

A vector field on $\mathbb{R}^3$ is a map $V : \mathbb{R}^3 \to T(\mathbb{R}^3)$ such that for $\mathbf{p} \in \mathbb{R}^3$, $V(\mathbf{p}) \in T_{\mathbf{p}}(\mathbb{R}^3)$. Then we can define $V[f] : \mathbb{R}^3 \to \mathbb{R}$, $\mathbf{p} \mapsto V(\mathbf{p})\,[f]$.

Alternatively given a vector field W on $\mathbb{R}^3$ and $\mathbf{v_p} \in T(\mathbb{R}^3)$ we could follow the definition of $\mathbf{v_p}\,[f]$ and define $\mathbf{v_p}[W] = W'(\mathbf{p} + t\mathbf{v})|_{t=0} \in T_{\mathbf{p}}(\mathbb{R}^3)$. In fact, this is denoted by $\nabla_{\mathbf{v_p}}[W]$.

Now we have the definition of $\nabla_{\mathbf{v_p}}[W]$ given V, another vector space on $\mathbb{R}^3$, define $\nabla_V[W] : \mathbb{R}^3 \to T(\mathbb{R}^3)$ by $\nabla_V[W]\,(\mathbf{p}) = \nabla_{V(\mathbf{p})}\,[W]$.

All the quantities measure the initial rate of change of either a scalar -valued function or vector field as you leave a point $\mathbf{p}$ in direction $\mathbf{v}$. The direction will either be given or will be the value of some vector field at $\mathbf{p}$.

1-forms

Definition 16 *A 1-**form** ϕ on $\mathbb{R}^3$ is a real-valued function on the set of all tangent vectors to $\mathbb{R}^3$such that ϕ is linear at each point of $\mathbb{R}^3$, that is*

$$\phi(a\,\mathbf{v_p} + b\,\mathbf{w_p}) = a\,\phi(\mathbf{v_p}) + b\,\phi(\mathbf{w_p})\,,$$

for all $a, b \in \mathbb{R}$, $\mathbf{v_p}, \mathbf{w_p} \in T_{\mathbf{p}}(\mathbb{R}^3)$ for all $\mathbf{p} \in \mathbb{R}^3$.

So $\phi : T(\mathbb{R}^3) \to \mathbb{R}$. On an initial glance this appears different to the differential 1-form defined in the Calculus of Several Variables course as a function $\boldsymbol{\omega} : U \subseteq \mathbb{R}^n \to \mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R})$. But to evaluate $\boldsymbol{\omega}$ in CoSV we need take $\mathbf{p} \in U$ and $\mathbf{v} \in \mathbb{R}^n$ when $\boldsymbol{\omega}_{\mathbf{p}}(\mathbf{v})$ can then be calculated. We can say that $\boldsymbol{\omega}$ depends on, is a function of, $\mathbf{v_p}$. For this reason, given $\boldsymbol{\omega}$ satisfying the CoSV definition of a 1-form define $\widetilde{\boldsymbol{\omega}} : T(\mathbb{R}^3) \to \mathbb{R}$ by $\widetilde{\boldsymbol{\omega}}(\mathbf{v_p}) = \boldsymbol{\omega}_{\mathbf{p}}(\mathbf{v}) \in \mathbb{R}$ for all $\mathbf{v_p} \in T(\mathbb{R}^3)$. This new function $\widetilde{\boldsymbol{\omega}}$ is linear because of the definition of $\mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R})$ as the set of linear functions and so $\widetilde{\boldsymbol{\omega}}$ satisfies definition 16.

Note from above that, given the differentiable $f : \mathbb{R}^3 \to \mathbb{R}$, the derivative $\mathbf{v_p}[f]$ is linear in that $\mathbf{v_p}[af + bg] = a\,\mathbf{v_p}[f] + b\,\mathbf{v_p}[g]$. So we have an example of a 1-form satisfying definition 16 in the following.

Definition 17 *Given the differentiable $f : \mathbb{R}^3 \to \mathbb{R}$,the **differential** 1-**form** df is defined by*

$$df(\mathbf{v_p}) = \mathbf{v_p}[f]\,,$$

for all tangent vectors $\mathbf{v_p}$.

This can be shown to have all the properties seen in the CoSV course. For example, label the projection functions as $x^i : \mathbb{R}^3 \to \mathbb{R}$, $\mathbf{x} \mapsto x^i$, $i = 1, 2$ or 3. Then, for $\mathbf{v_p} \in T(\mathbb{R}^3)$,

$$dx^i(\mathbf{v_p}) = \frac{d}{dt} x^i(\mathbf{p} + t\mathbf{v})\bigg|_{t=0} = \frac{d}{dt}\big(p^i + tv^i\big)\bigg|_{t=0} = v^i,$$

for $i = 1, 2$ and 3.

Given a 1-form ϕ (satisfying definition 16) write $\mathbf{v_p} = \sum_{j=1}^{3} v^j\, U_j(\mathbf{p})$ so, by linearity,

$$\phi(\mathbf{v_p}) = \sum_{j=1}^{3} v^j\, \phi\left(U_j(\mathbf{p})\right) = \sum_{j=1}^{3} \phi_j(\mathbf{v_p})\, dx^i(\mathbf{v_p})$$

where $\phi_j : T(\mathbb{R}^3) \to \mathbb{R}$ is defined by $\mathbf{v_p} \mapsto \phi\,(U_j(\mathbf{p}))$. Thus we can write $\phi = \sum_{j=1}^{3} \phi_j \, dx^i$.

If $\phi = df$ for some differentiable f then

$$df(U_j(\mathbf{p})) = \frac{d}{dt} f\,(\mathbf{p} + t\,U_i)\Big|_{t=0} = \frac{\partial}{\partial x^i} f(\mathbf{p})\,.$$

and so

$$df(\mathbf{v_p}) = \sum_{j=1}^{3} v^j \, df(U_j(\mathbf{p})) = \sum_{j=1}^{3} \frac{\partial}{\partial x^i} f(\mathbf{p})\, dx^i(\mathbf{v_p})\,.$$

So we can write

$$df = \sum_{j=1}^{3} \frac{\partial f}{\partial x^i} dx^i.$$

Jacobian Matrix

Definition 18 *Let* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$ *be a mapping. Then define the* ***derivative map*** $\mathbf{F}_*$ *on* $T\,(\mathbb{R}^n)$ *as follows. Given* $\mathbf{v} \in T\,(\mathbb{R}^n)$ *there exists* $\mathbf{p} \in \mathbb{R}^n$ *such that* $\mathbf{v} = \mathbf{v_p} \in T_{\mathbf{p}}\,(\mathbb{R}^n)$*. Then set*

$$\mathbf{F}_*\,(\mathbf{v}) = \frac{d}{dt}\mathbf{F}(\mathbf{p} + t\mathbf{v})\Big|_{t=0}.$$

This is the derivative of a curve $t \mapsto \mathbf{F}(\mathbf{p} + t\mathbf{v})$ and so a tangent vector (with point of application $\mathbf{F}(\mathbf{p})$). Thus $\mathbf{F}_* : T\,(\mathbb{R}^n) \to T\,(\mathbb{R}^m)$.

We cannot say that $\mathbf{F}_*$ is linear since we can only add together vectors with the same point of application. But we can restrict $\mathbf{F}_*$ to $T_{\mathbf{p}}\,(\mathbb{R}^n)$, getting the map $\mathbf{F}_{*\mathbf{p}} : T_{\mathbf{p}}\,(\mathbb{R}^n) \to T_{\mathbf{F}(\mathbf{p})}\,(\mathbb{R}^m)$. This map is linear and the matrix associated with this linear map is the **Jacobian matrix of F at p**.